Appendix A. Linear approximation of the stock price formula

Stock price dynamics follows the standard formula:

$$P_t = \mathbb{E}_t(M_{t+1}P_{t+1}) + \mathbb{E}_t(M_{t+1}D_{t+1}),$$

or

$$P_t = p_t = \mathbb{E}_t\left(\exp(m_{t+1} + p_{t+1})\right) + \mathbb{E}_t\left(\exp(m_{t+1} + d_{t+1})\right),$$

where small letters denote log variables. From the loglinear solution of the macroeconomic part of the model, dynamics of the stochastic discount factor and the dividends has the following state space form:

$$x_t = \bar{x} + M_x\hat{s}_{t-1} + W_x\epsilon_t, \quad \hat{s}_t = M\hat{s}_{t-1} + W\epsilon_t, \quad \epsilon_t \sim N(0, \Sigma),$$

where x_t represents $\log(M_t)$ or $\log(D_t)$. Because for the lognormal random variable $\mathbb{E}(\exp(x)) = \exp(\mathbb{E}(x) + 0.5\mathbb{D}^2(x))$, then we have:

$$p_t = \ln\left[\exp\left(\mathbb{E}_t(m_{t+1} + p_{t+1}) + 0.5\mathbb{D}_t^2(m_{t+1} + p_{t+1})\right) + \exp\left(\mathbb{E}_t(m_{t+1} + d_{t+1}) + 0.5\mathbb{D}_t^2(m_{t+1} + d_{t+1})\right)\right]. \tag{A.1}$$

We can approximate the above formula further using the fact that:

$$\ln\left(\exp(x) + \exp(y)\right) \approx \ln\left(X_0 + Y_0\right) + \frac{X_0}{X_0 + Y_0}(x - x_0) + \frac{Y_0}{X_0 + Y_0}(y - y_0), \tag{A.2}$$

where $X_0 = \exp(x_0)$ and $Y_0 = \exp(y_0)$. As a result, we get:

$$\begin{aligned} p_t =& \ln\left(A + B\right) + \\ &+ \frac{A}{A+B}\left(\mathbb{E}_t(m_{t+1} + p_{t+1}) + 0.5\mathbb{D}_t^2(m_{t+1} + p_{t+1}) - \ln(A)\right) + \\ &+ \frac{B}{A+B}\left(\mathbb{E}_t(m_{t+1} + d_{t+1}) + 0.5\mathbb{D}_t^2(m_{t+1} + d_{t+1}) - \ln(B)\right) = \\ =& \ln\left(A + B\right) + \\ &+ \frac{A}{A+B}\left(\mathbb{E}_t(m_{t+1} + p_{t+1}) - \mathbb{E}(m_{t+1} + p_{t+1})\right) + \\ &+ \frac{B}{A+B}\left(\mathbb{E}_t(m_{t+1} + d_{t+1}) - \mathbb{E}(m_{t+1} + d_{t+1})\right), \end{aligned} \tag{A.3}$$

where $A = \exp[\mathbb{E}(m_{t+1} + p_{t+1}) + 0.5\mathbb{D}_t^2(m_{t+1} + p_{t+1})]$, $B = \exp[\mathbb{E}(m_{t+1} + d_{t+1}) + 0.5\mathbb{D}_t^2(m_{t+1} + d_{t+1})]$.

Appendix B. Stock price dynamics under rational expectations

Assuming linear law of motion for m_t, d_t and p_t:

$$m_t = \bar{m} + M_m \hat{s}_{t-1} + W_m \epsilon_t, \quad d_t = \bar{d} + M_d \hat{s}_{t-1} + W_d \epsilon_t, \quad p_t = \bar{p} + M_p \hat{s}_{t-1} + W_p \epsilon_t, \tag{B.4}$$

the dynamics of stock price has the following form:

$$\begin{aligned} p_t =& \ln(A+B) + \frac{A}{A+B}(M_m + M_p)\hat{s}_t + \frac{B}{A+B}(M_m + M_d)\hat{s}_t = \\ =& \ln(A+B) + \left(M_m + \frac{A}{A+B}M_p + \frac{B}{A+B}M_d\right)\hat{s}_t = \\ =& \ln(A+B) + \left(M_m + \frac{A}{A+B}M_p + \frac{B}{A+B}M_d\right)M\hat{s}_{t-1} + \\ &+ \left(M_m + \frac{A}{A+B}M_p + \frac{B}{A+B}M_d\right)W\epsilon_t, \end{aligned} \tag{B.5}$$

where $A = \exp\left[\bar{m} + \bar{p} + 0.5(W_m + W_p)\Sigma(W_m + W_p)'\right]$, $B = \exp\left[\bar{m} + \bar{d} + 0.5(W_m + W_d)\Sigma(W_m + W_d)'\right]$. Therefore, we obtain the following mapping from the PLM to the ALM coefficients:

$$T\begin{pmatrix} \bar{p} \\ M_p \\ W_p \end{pmatrix} = \begin{pmatrix} \ln(A+B) \\ \left(M_m + \frac{A}{A+B}M_p + \frac{B}{A+B}M_d\right)M \\ \left(M_m + \frac{A}{A+B}M_p + \frac{B}{A+B}M_d\right)W \end{pmatrix} \tag{B.6}$$

Appendix C. Jermann model with adaptive learning for macroeconomic state variables

Introducing the state variables learning in the model we follow Carceles-Poveda and Giannitsarou (2007). For brevity, we consider the model with the state variables only. However, taking into account clarity of the presentation, we start with six variables: consumption C, capital K, technological shock Z, investment I, price of capital Q and marginal rate of substitution M. Hence, the model consists of six equations:

$$Q_t = (\gamma - 1 + \delta)^{-1/\xi} \left(\frac{\gamma \widetilde{I}_t}{\widetilde{K}_t} \right)^{1/\xi}, \tag{C.7}$$

$$Q_t = \mathbb{E}_t \left[M_{t+1} \left(\alpha Z_{t+1} \left(\frac{\widetilde{K}_{t+1}}{\gamma} \right)^{\alpha - 1} - \frac{\gamma \widetilde{I}_{t+1}}{\widetilde{K}_{t+1}} + Q_{t+1} \left[1 - \delta + \Phi \left(\frac{\gamma \widetilde{I}_{t+1}}{\widetilde{K}_{t+1}} \right) \right] \right) \right], \tag{C.8}$$

$$M_t = \frac{\widetilde{\beta}}{\gamma} \frac{\left(\widetilde{C}_t - \frac{\chi_c}{\gamma} \widetilde{C}_{t-1} \right)^{-\nu} - \chi_c \frac{\widetilde{\beta}}{\gamma} \mathbb{E}_t \left[\left(\widetilde{C}_{t+1} - \frac{\chi_c}{\gamma} \widetilde{C}_t \right)^{-\nu} \right]}{\left(\widetilde{C}_{t-1} - \frac{\chi_c}{\gamma} \widetilde{C}_{t-2} \right)^{-\nu} - \chi_c \frac{\widetilde{\beta}}{\gamma} \mathbb{E}_{t-1} \left[\left(\widetilde{C}_t - \frac{\chi_c}{\gamma} \widetilde{C}_{t-1} \right)^{-\nu} \right]}, \tag{C.9}$$

$$\gamma K_t = \left[1 - \delta + \Phi \left(\frac{\gamma I_t}{K_t} \right) \right] K_{t-1}, \tag{C.10}$$

$$Z_t \left(\frac{\widetilde{K}_t}{\gamma} \right)^{\alpha} = \widetilde{C}_t + \widetilde{I}_t, \tag{C.11}$$

$$Z_t = \rho_z Z_{t-1} + \sigma_z \epsilon_t. \tag{C.12}$$

In the first step, we loglinearize the equations and then substitute out the non-state variables I, Q and M. Next, we rewrite the system in the reduced form. Finally, we derive the actual law of motion of the state variables under rational expectations and adaptive learning.

The steady state values of the six analysed variables are as follows:

$$\begin{aligned}
\bar{K} &= \gamma \left(\frac{\alpha \widetilde{\beta}}{\gamma - \widetilde{\beta}(1 - \delta)} \right)^{\frac{1}{1-\alpha}}, \\
\bar{C} &= \left(\frac{\bar{K}}{\gamma} \right)^{\alpha} - \frac{\gamma - 1 + \delta}{\gamma} \bar{K}, \\
\bar{Z} &= 0, \\
\bar{I} &= \frac{\gamma - 1 + \delta}{\gamma} \bar{K}, \\
\bar{M} &= \frac{\widetilde{\beta}}{\gamma}, \\
\bar{Q} &= 1.
\end{aligned}$$

Loglinearizing the system (C.7)–(C.12), we obtain the following set of linear difference equations:

$$0 = a_{11}\hat{i}_t + a_{12}\hat{q}_t + a_{13}\hat{k}_{t-1}, \tag{C.13}$$

$$0 = a_{21}\mathbb{E}_t\hat{m}_{t+1} + a_{22}\mathbb{E}_t Z_{t+1} + a_{23}\mathbb{E}_t\hat{q}_{t+1} + a_{24}\mathbb{E}_t\hat{i}_{t+1} + a_{25}\hat{k}_t + a_{26}\hat{q}_t, \tag{C.14}$$

$$0 = a_{31}\mathbb{E}_t\hat{c}_{t+1} + a_{32}\hat{c}_t + a_{33}\hat{m}_t + a_{34}\hat{c}_{t-1} + a_{35}\hat{c}_{t-2}, \tag{C.15}$$

$$0 = a_{41}\hat{i}_t + a_{42}\hat{k}_t + a_{43}\hat{k}_{t-1}, \tag{C.16}$$

$$0 = a_{51}\hat{c}_t + a_{52}\hat{i}_t + a_{53}Z_t + a_{54}\hat{k}_{t-1}, \tag{C.17}$$

$$0 = \rho_z Z_{t-1} + \sigma_z\epsilon_t - Z_t, \tag{C.18}$$

where:

$$a_{11} = \frac{1}{\xi}, \qquad a_{12} = 1, \qquad a_{13} = -\frac{1}{\xi},$$

$$a_{21} = \frac{\widetilde{\beta}}{\gamma}\left[\alpha\left(\frac{\bar{K}}{\gamma}\right)^{\alpha-1} - \delta + 1\right], \qquad a_{22} = \frac{\widetilde{\beta}}{\gamma}\alpha\left(\frac{\bar{K}}{\gamma}\right)^{\alpha-1}, \qquad a_{23} = \widetilde{\beta},$$

$$a_{24} = 0, \qquad a_{25} = -\frac{\widetilde{\beta}}{\gamma}\alpha(1-\alpha)\left(\frac{\bar{K}}{\gamma}\right)^{\alpha-1}, \qquad a_{26} = -1,$$

$$a_{31} = \frac{\widetilde{\beta}^2\nu\chi}{(\gamma-\chi)(\gamma-\widetilde{\beta}\chi)}, \qquad a_{32} = -\frac{\widetilde{\beta}\nu\chi(1+2\widetilde{\beta})}{(\gamma-\chi)(\gamma-\widetilde{\beta}\chi)} - \frac{\widetilde{\beta}\nu}{\gamma}, \qquad a_{33} = -\frac{\widetilde{\beta}}{\gamma},$$

$$a_{34} = \frac{\widetilde{\beta}\nu\chi(2+\widetilde{\beta})}{(\gamma-\chi)(\gamma-\widetilde{\beta}\chi)} + \frac{\widetilde{\beta}\nu}{\gamma}, \qquad a_{35} = -\frac{\widetilde{\beta}\nu\chi}{(\gamma-\chi)(\gamma-\widetilde{\beta}\chi)},$$

$$a_{41} = \bar{I}, \qquad a_{42} = -\gamma\bar{K}, \qquad a_{43} = (1-\delta)\bar{K},$$

$$a_{51} = \bar{C}, \qquad a_{52} = \bar{I}, \qquad a_{53} = -\left(\frac{\bar{K}}{\gamma}\right)^{\alpha},$$

$$a_{54} = -\alpha\left(\frac{\bar{K}}{\gamma}\right)^{\alpha}.$$

Then, we substitute out q, m and i. As a result, equations (C.14) and (C.17) can be rewritten as:

$$0 = b_{11}\mathbb{E}_t\hat{c}_{t+2} + b_{12}\mathbb{E}_t\hat{c}_{t+1} + b_{13}\mathbb{E}_t Z_{t+1} + b_{14}\mathbb{E}_t\hat{k}_{t+1} + b_{15}\hat{c}_t + b_{16}\hat{k}_t + b_{17}\hat{c}_{t-1} + b_{18}\hat{k}_{t-1}, \tag{C.19}$$

$$0 = b_{21}\hat{c}_t + b_{22}\hat{k}_t + b_{23}\hat{k}_{t-1} + b_{24}Z_t, \tag{C.20}$$

where:

$$b_{11} = -a_{21}\frac{a_{31}}{a_{33}}, \qquad b_{12} = a_{21}\frac{a_{32}}{a_{33}}, \qquad b_{13} = a_{22}, \qquad b_{14} = \frac{a_{42}}{a_{41}}\left(-a_{24} + a_{23}\frac{a_{11}}{a_{12}}\right), \qquad b_{15} = -a_{21}\frac{a_{34}}{a_{33}},$$
$$b_{16} = \left(a_{25} - a_{24}\frac{a_{43}}{a_{41}} + a_{23}\frac{a_{11}a_{43}}{a_{12}a_{41}} - a_{23}\frac{a_{13}}{a_{12}} + a_{26}\frac{a_{11}a_{42}}{a_{12}a_{41}}\right), \qquad b_{17} = -a_{21}\frac{a_{35}}{a_{33}}, \qquad b_{18} = a_{26}\left(\frac{a_{11}a_{43}}{a_{12}a_{41}} - \frac{a_{13}}{a_{12}}\right),$$
$$b_{21} = a_{51}, \qquad b_{22} = -a_{52}\frac{a_{42}}{a_{41}}, \qquad b_{23} = \left(a_{54} - a_{52}\frac{a_{43}}{a_{41}}\right), \qquad b_{24} = a_{53}.$$

Now, we want to solve the system for $\hat{c}_t$ and $\hat{k}_t$. From (C.20), we have:

$$\hat{k}_t = -\frac{b_{21}}{b_{22}}\hat{c}_t - \frac{b_{23}}{b_{22}}\hat{k}_{t-1} - \frac{b_{24}}{b_{22}}Z_t, \tag{C.21}$$

and therefore:

$$\begin{aligned}\mathbb{E}_t\hat{k}_{t+1} &= -\frac{b_{21}}{b_{22}}\mathbb{E}_t\hat{c}_{t+1} - \frac{b_{23}}{b_{22}}\hat{k}_t - \frac{b_{24}}{b_{22}}\mathbb{E}_tZ_{t+1} = -\frac{b_{21}}{b_{22}}\mathbb{E}_t\hat{c}_{t+1} - \frac{b_{23}}{b_{22}}\left(-\frac{b_{21}}{b_{22}}\hat{c}_t - \frac{b_{23}}{b_{22}}\hat{k}_{t-1} - \frac{b_{24}}{b_{22}}Z_t\right) - \frac{b_{24}}{b_{22}}\mathbb{E}_tZ_{t+1} = \\ &= -\frac{b_{21}}{b_{22}}\mathbb{E}_t\hat{c}_{t+1} + \frac{b_{23}b_{21}}{b_{22}b_{22}}\hat{c}_t + \left(\frac{b_{23}}{b_{22}}\right)^2\hat{k}_{t-1} + \frac{b_{23}b_{24}}{b_{22}b_{22}}Z_t - \frac{b_{24}}{b_{22}}\mathbb{E}_tZ_{t+1}.\end{aligned} \tag{C.22}$$

Plugging (C.21) and (C.22) into (C.19) and using the fact that $\mathbb{E}_tZ_{t+1} = \rho_z Z_t$ gives:

$$\begin{aligned}0 = b_{11}\mathbb{E}_t\hat{c}_{t+2} &+ \left(b_{12} - b_{14}\frac{b_{21}}{b_{22}}\right)\mathbb{E}_t\hat{c}_{t+1} + \left(b_{15} + b_{14}\frac{b_{23}b_{21}}{b_{22}b_{22}} - b_{16}\frac{b_{21}}{b_{22}}\right)\hat{c}_t + b_{17}\hat{c}_{t-1} + \\ &+ \left(b_{18} + b_{14}\left(\frac{b_{23}}{b_{22}}\right)^2 - b_{16}\frac{b_{23}}{b_{12}}\right)\hat{k}_{t-1} + \left[\left(b_{13} - b_{14}\frac{b_{24}}{b_{22}}\right)\rho_z + b_{14}\frac{b_{23}b_{24}}{b_{22}b_{22}} - b_{16}\frac{b_{24}}{b_{22}}\right]Z_t.\end{aligned} \tag{C.23}$$

Hence:

$$\begin{aligned}\hat{c}_t = \frac{b_{11}}{b_c}\mathbb{E}_t\hat{c}_{t+2} &+ \frac{1}{b_c}\left(b_{12} - b_{14}\frac{b_{21}}{b_{22}}\right)\mathbb{E}_t\hat{c}_{t+1} + \frac{b_{17}}{b_c}\hat{c}_{t-1} + \frac{1}{b_c}\left(b_{18} + b_{14}\left(\frac{b_{23}}{b_{22}}\right)^2 - b_{16}\frac{b_{23}}{b_{12}}\right)\hat{k}_{t-1} + \\ &+ \frac{1}{b_c}\left[\left(b_{13} - b_{14}\frac{b_{24}}{b_{22}}\right)\rho_z + b_{14}\frac{b_{23}b_{24}}{b_{22}b_{22}} - b_{16}\frac{b_{24}}{b_{22}}\right]Z_t, \qquad b_c = -\left(b_{15} + b_{14}\frac{b_{23}b_{21}}{b_{22}b_{22}} - b_{16}\frac{b_{21}}{b_{22}}\right).\end{aligned} \tag{C.24}$$

Finally, plugging (C.24) into (C.21), we obtain:

$$\begin{aligned}\hat{k}_t = &-\frac{b_{11}b_{21}}{b_cb_{22}}\mathbb{E}_t\hat{c}_{t+2} - \frac{b_{21}}{b_cb_{22}}\left(b_{12} - b_{14}\frac{b_{21}}{b_{22}}\right)\mathbb{E}_t\hat{c}_{t+1} - \frac{b_{17}b_{21}}{b_cb_{22}}\hat{c}_{t-1} - \\ &-\left[\frac{b_{21}}{b_cb_{22}}\left(b_{18} + b_{14}\left(\frac{b_{23}}{b_{22}}\right)^2 - b_{16}\frac{b_{23}}{b_{12}}\right) + \frac{b_{23}}{b_{22}}\right]\hat{k}_{t-1} + \\ &-\left[\frac{b_{21}}{b_cb_{22}}\left(\left(b_{13} - b_{14}\frac{b_{24}}{b_{22}}\right)\rho_z + b_{14}\frac{b_{23}b_{24}}{b_{22}b_{22}} - b_{16}\frac{b_{24}}{b_{22}}\right) + \frac{b_{24}}{b_{22}}\right]Z_t.\end{aligned} \tag{C.25}$$

Summarizing, the model can be compactly written as follows:

$$\hat{c}_t = \phi_{11}\mathbb{E}_t\hat{c}_{t+2} + \phi_{12}\mathbb{E}_t\hat{c}_{t+1} + \phi_{13}\hat{c}_{t-1} + \phi_{14}\hat{k}_{t-1} + \phi_{15}Z_t,$$
$$\hat{k}_t = \phi_{21}\mathbb{E}_t\hat{c}_{t+2} + \phi_{22}\mathbb{E}_t\hat{c}_{t+1} + \phi_{23}\hat{c}_{t-1} + \phi_{24}\hat{k}_{t-1} + \phi_{25}Z_t,$$
$$Z_t = \rho_z Z_{t-1} + \sigma_z\epsilon_t,$$

or in matrix form:

$$\hat{x}_t = \Phi_{11}\mathbb{E}_t\hat{x}_{t+2} + \Phi_{12}\mathbb{E}_t\hat{x}_{t+1} + \Phi_{13}\hat{x}_{t-1} + \Phi_{14}Z_t, \tag{C.26}$$
$$Z_t = \rho_z Z_{t-1} + \sigma_z\epsilon_t, \tag{C.27}$$

where $\hat{x}_t = [\hat{c}_t \quad \hat{k}_t]^T$ represents vector of the state variables (without the technological shock Z_t).

The RE solution of the model has the following form:

$$\hat{x}_t = \psi_x^{RE}\hat{x}_{t-1} + \psi_z^{RE}Z_{t-1} + \psi_\epsilon^{RE}\epsilon_t. \tag{C.28}$$

In the adaptive learning case, we assume that agents know the true functional form of the solution (C.28), but do not have knowledge on the values of the parameters. Moreover, they correctly know both the functional form as well as the parameter values of the law of motion of the technology shock (C.27). As a result, the perceived law of motion of the state variables is given by:

$$\hat{x}_t = \psi_x^{AL}\hat{x}_{t-1} + \psi_z^{AL}Z_{t-1} + u_t, \tag{C.29}$$

where ψ_x^{AL} and ψ_z^{AL} represents sample estimates of the matrices ψ_x^{RE} and ψ_z^{RE}. Similarly to the stock price learning case, we employ the constant gain recursive least squares algorithm for estimating these matrices.

Because:

$$\mathbb{E}_t\hat{x}_{t+1} = \psi_x^{AL}\hat{x}_t + \psi_z^{AL}Z_t,$$
$$\mathbb{E}_t\hat{x}_{t+2} = \mathbb{E}_t\left[\psi_x^{AL}\hat{x}_{t+1} + \psi_z^{AL}Z_{t+1}\right] = \psi_x^{AL}\left(\psi_x^{AL}\hat{x}_t + \psi_z^{AL}Z_t\right) + \rho_z\psi_z^{AL}Z_t,$$

then the dynamics of the state variables (C.26) under adaptive learning can be written as:

$$
\begin{aligned}
\hat{x}_t &= \Phi_{11}\left[\left(\psi_x^{AL}\right)^2 \hat{x}_t + \psi_x^{AL}\psi_z^{AL} Z_t + \rho_z \psi_z^{AL} Z_t\right] + \\
&\quad + \Phi_{12}\left[\psi_x^{AL}\hat{x}_t + \psi_z^{AL} Z_t\right] + \Phi_{13}\hat{x}_{t-1} + \Phi_{14} Z_t, \\
\left[\mathrm{I} - \Phi_{11}\left(\psi_x^{AL}\right)^2 - \Phi_{12}\psi_x^{AL}\right]\hat{x}_t &= \Phi_{13}\hat{x}_{t-1} + \left[\Phi_{11}\left(\psi_x^{AL}\psi_z^{AL} + \rho_z\psi_z^{AL}\right) + \Phi_{12}\Psi_z^{AL} + \Phi_{14}\right] Z_t, \\
\hat{x}_t &= \Psi_1^{-1}\Phi_{13}\hat{x}_{t-1} + \Psi_1^{-1}\Psi_2 Z_t, \qquad \text{(C.30)}
\end{aligned}
$$

where:

$$
\Psi_1 = \mathrm{I} - \Phi_{11}\left(\psi_x^{AL}\right)^2 - \Phi_{12}\psi_x^{AL}, \qquad \Psi_2 = \Phi_{11}\left(\psi_x^{AL}\psi_z^{AL} + \rho_z\psi_z^{AL}\right) + \Phi_{12}\Psi_z^{AL} + \Phi_{14}.
$$

Equation (C.30) describes the actual law of motion for the state variables under the adaptive learning scheme. Dynamics of all other variables in the model is determined by the evolution of the state variables.

References

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