Adaptive learning and asset prices

in production economies

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Abstract: We investigate to what extent adaptive learning affects asset prices in general equilibrium models with production sector. We introduce the adaptive learning assumption into the stock price equation only and assume that agents learn on the next period stock price. We also propose a novel, linear version of the loglinear lognormal stock price formula. Using the standard growth model with habit formation and convex investment costs, we show that constant gain adaptive learning significantly increases the equity premium as well as excess returns predictability and therefore helps to match the basic macro-financial stylized facts. These results are primarily driven by learning on the long-run mean level of stock prices. However, extending the learning scheme to macroeconomic state variables partially offset these gains.

1 Introduction

Adaptive learning mechanisms are proposed as an explanation for some important asset pricing facts like high equity premium, excess volatility of stock returns, momentum or excess returns predictability. However, usually simple endowment economy models are employed to assess quantitatively to what extent the adaptive learning schemes are able to explain the mentioned effects. On the other hand, the existing results that accounts for the production sector (Carceles-Poveda and Giannitsarou, 2008) are based on the simplest stochastic growth model. Unfortunately, because the asset price implications of the rational expectations version of this model are in sharp contrast to empirical data, introducing adaptive learning mechanism cannot solely resolve all the problems. In this paper, we embed the adaptive learning scheme into more comprehensive models that are better suited for explaining asset price phenomena. We show that adaptive learning significantly amplifies the effects generated by other mechanisms which considerably facilitate matching various stylized facts of asset price data.

In the baseline version of our study, we use the model proposed by Jermann (1998). It extends the standard stochastic growth model to account for consumption habits in the utility function and convex investment costs. The model in the rational expectations version is able to generate high equity premium and match the basic characteristics of macroeconomic variables. However, it fails to explain excess returns predictability. For the robustness check purposes, we also consider the standard stochastic growth model as well as the slow-moving habits model proposed by Jaccard (2014). The latter model can be viewed as a generalization of the Jermann model that allows for the long-memory habit formation process.

The important contribution of the paper is a new formula for approximating stock price dynamics —which is a linearized version of the loglinear lognormal approximate equation proposed by Jermann (1998). To derive it, we first loglinearize macroeconomic part of a model. Then, we employ the loglinear lognormal formula for the one-period risk free rate as well as the stock price. Finally, we linearize the formula for the stock price around its risk-adjusted mean. We show that stock price characteristics obtained with the proposed method are usually very close to their counterparts from the third order perturbation approximation. With the linear law of motion for stock prices, we

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are able to use the standard linear learning rules. Moreover, we can disentangle the effects of learning on the mean level of stock prices, their dynamics with respect to the state variables and the price of risk. There is also consistency between the functional forms of all the components of the stock price equation — the pricing kernel, dividends and stock prices.

The analysis of quantitative consequences of replacing the rational expectations hypothesis with the adaptive learning assumption for asset prices in production economies is important at least for two reasons. First, because under adaptive learning there could be significant differences between asset price properties of models with and without the production sector. They can mainly stem from the differences in amount and properties of information that agents can utilize during the learning process. In endowment economies, agents are usually assumed to use only the exogenous dividend process as the explanatory variable whereas in more realistic models with the production sectors they can utilize the whole set of endogenously determined state variables. Moreover, in production economies, we can extend the learning assumption to the macroeconomic state variables, which can also seriously affect implied asset price characteristics of a model and, as shown in our study, partially offset the impact of stock price learning.

The second argument comes from the fact that implied asset price moments are used to examine reliability of macroeconomic models (see Cochrane 2007, p. 242; Fernández-Villaverde 2009, p. 42). The most famous example of such attitude is the equity premium puzzle presented by Mehra and Prescott (1985). According to the macro-finance literature, asset prices and macroeconomic variables are generally driven by the same factors. From this point of view, the spectacular fail of the workhorse stochastic growth model in generating any significant equity premium puts serious doubts on its ability to explain basic macroeconomic stylized facts correctly.

Our work is closely related to the study of Carceles-Poveda and Giannitsarou (2008) who analyze to what extent adaptive learning can explain various asset pricing facts in the simple stochastic growth model. They find that, in absolute terms, introducing the adaptive learning mechanism results in minor changes in the asset pricing properties of the model compared to its rational expectations version. We conduct similar analysis but our work differs in few important points. First, as already mentioned, we

work with the richer model proposed by Jermann. Furthermore, Carceles-Poveda and Giannitsarou (2008) assume that agents correctly know the mean stock price and learn only on deviations from the steady state. We allow agents to learn also on the long-run mean of stock price and show that it generates significant asset price effects in their model. And finally, agents in their setup learn simultaneously on both, stock prices and macroeconomic variables, while we disentangle these two effects and study them separately.

This paper can also be seen as an extension of the study conducted by Adam et al. (2012) to the production economies. They assume that agents learn on the next period's stock price using the constant gain recursive least square scheme. In the endowment economy framework, this assumption is able to deliver sizable equity premium, momentum and excess returns predictability. We confirm these findings considering exactly the same type of learning while working with the more realistic economies with the production sector.

There are also many other papers that study asset price implications of the adaptive learning models. Closely related to our study is the paper by Bullard and Duffy (2001) who employ a life-cycle model with capital accumulation to show that learning on stock prices can explain excess volatility of stock prices. Similar results are reported by Timmermann (1993, 1996). These early studies use simple partial equilibrium models with Bayesian learning on the dividend process. Brennan and Xia (2001), Cogley and Sargent (2008) and Cecchetti et al. (2000), among others, extend the framework to pure exchange general equilibrium models. They document that Bayesian learning on the endowment process generates plausible level of the equity premium. Comprehensive study of the impact of Bayesian learning on various model's parameters is also carried out by Collin-Dufresne et al. (2013).

Another important strand of the literature aims at explaining the asset price phenomena with heterogeneous learning rules employed by agents. We should mentioned here the seminal paper by Brock and Hommes (1998), as well as the works of Branch and Evans (2010) and Lewis and Markiewicz (2009) to name but a few. Contrary to the previous studies, the last paper solves the exchange rate disconnect puzzle using misspecified learning schemes. In the same vein, the learning mechanism is proposed

as the explanation of the forward premium puzzle (Chakraborty and Evans, 2008) and housing price dynamics in the recent years (Gelain and Lansing, 2014).

Finally, our paper contributes to the literature on asset price approximation in macroeconomic models. General algorithms for calculating the risk-adjusted linear approximations of DSGE models are developed by Malkhozov (2014) and Meyer-Gohde (2014). Unfortunately, they are significantly more demanding computationally compared to our closed form solution for asset prices. Higher order generalizations of the log-linear lognormal approximation for bond prices are discussed in Andreasen and Żabczyk (2015) among others.

The rest of the paper is organized as follows. First, we introduce our linear stock price approximation method. Then, we present the adaptive learning mechanism employed in the paper. Section four contains description of the models. Finally, in the last two sections, we present results of the simulation studies for the baseline model and a few robustness check exercises.

2 Linear approximation of the stock price equation

Because it is much easier to study learning algorithms within a linear setup, we need a risk-adjusted linear law of motion for stock prices. Our approximation procedure consists of two steps. First, we employ the lognormal approximation, similarly to Jermann (1998). Then, the resulted formula is linearized around the risk-adjusted mean. As a result, we get a difference functional equation for stock prices with a linear solution.

Let P_t be the stock price in period t, M_t denotes the one-period-ahead stochastic discount factor, D_t stands for the dividend and $\mathbb{E}_t^*$ is the conditional expectation (not necessarily mathematical) operator. Stock price dynamics is described by the well-known equation:

$$P_t = \mathbb{E}_t^*(M_{t+1}P_{t+1}) + \mathbb{E}_t^*(M_{t+1}D_{t+1}). \tag{1}$$

Relationship (1) can also be approximately written as follows:

$$
\begin{aligned}
p_t = \log \big[&\exp \left(\mathbb{E}_t^*(m_{t+1} + p_{t+1}) + 0.5\mathbb{D}_t^{2*}(m_{t+1} + p_{t+1}) \right) + \\
&+ \exp \left(\mathbb{E}_t^*(m_{t+1} + d_{t+1}) + 0.5\mathbb{D}_t^{2*}(m_{t+1} + d_{t+1}) \right) \big],
\end{aligned}
\tag{2}
$$

where small letters denote log variables and $\mathbb{D}_t^{2*}$ represents the conditional variance operator. Equation (2) is exact provided M_{t+1}, D_{t+1} and P_{t+1} are lognormally distributed.[1]

Because dynamics of M_{t+1} and D_{t+1} is endogenously determined by the macroeconomic part of the studied model these variables are indeed conditionally lognormal as long as we work with the loglinearized version of the macroeconomic model with the following state-space representation:

$$x_t = \bar{x} + M_x \hat{s}_{t-1} + W_x \epsilon_t, \quad \hat{s}_t = M \hat{s}_{t-1} + W \epsilon_t, \quad \epsilon_t \sim N(0, \Sigma), \tag{3}$$

where x_t represents $\log M_t$ or $\log D_t$, $\hat{s}_t$ denotes vector of state variables expressed as log-deviations from a steady state, ϵ_t is vector of stochastic shocks and M_x, W_x, M and W are matrices that govern dynamics of the system. This property, coupled with the discounted-dividend version of the stock price equation, was exploited by Jermann (1998) in his loglinear lognormal approximation for asset prices.

However, lognormality does not hold for P_{t+1}, even if it is defined by (2) with lognormal M_{t+1} and D_{t+1}. Therefore, we further simplify the stock price formula using linear approximations for exponential and logarithmic functions. As a result, we get the following decomposition:[2]

$$
\begin{aligned}
p_t = {} & \log{(A+B)} + \frac{A}{A+B}\left(\mathbb{E}_t^*(m_{t+1}+p_{t+1}) - \mathbb{E}^*(m_{t+1}+p_{t+1})\right) + \\
& + \frac{B}{A+B}\left(\mathbb{E}_t^*(m_{t+1}+d_{t+1}) - \mathbb{E}^*(m_{t+1}+d_{t+1})\right),
\end{aligned}
\tag{4}
$$

where:

$$A = \exp[\mathbb{E}^*(m_{t+1}+p_{t+1}) + 0.5\mathbb{D}_t^{2*}(m_{t+1}+p_{t+1})],$$

$$B = \exp[\mathbb{E}^*(m_{t+1}+d_{t+1}) + 0.5\mathbb{D}_t^{2*}(m_{t+1}+d_{t+1})].$$

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[1] We use the fact that if $x \sim N(\mu, \sigma)$, then $\mathbb{E}(\exp(x)) = \exp(\mu + 0.5\sigma^2)$.

[2] To obtain this formula, the following approximate relationship is applied:

$$\ln{(\exp(x) + \exp(y))} \approx \ln{(X_0 + Y_0)} + \frac{X_0}{X_0 + Y_0}(x - x_0) + \frac{Y_0}{X_0 + Y_0}(y - y_0)$$

where $X_0 = \exp(x_0)$ and $Y = \exp(y_0)$, taking A and B as the expansion points. For details, see the technical appendix.

This formula has a very intuitive form. The term $\log(A + B)$ represents the risk-adjusted long-run mean of stock prices, whereas the other two expressions can be viewed as the expected deviations of the discounted next-period stock price and dividend, respectively, from their long-run means.

In the rest of the paper, we assume that agents use formula (4) for pricing stock in both, rational expectations and adaptive learning, versions of the models. In the technical appendix, we show that assuming linear law of motion for the stochastic discount factor and dividends as in (3), actual dynamics of stock prices under rational expectations has the linear state space form similar to those for m_t and d_t. Further in the paper, we check the accuracy of the stock price formula. We show that it gives similar results to the third order perturbation method for the analyzed models.

In our study, we also need to calculate prices of a risk-free bond $P_{f,t}$. We do this using the loglinear lognormal approximation and obtain the following result:

$$p_{f,t} = \log \mathbb{E}_t M_{t+1} = \bar{m} + W_m \Sigma W_m' + M_m M \hat{s}_{t-1} + M_m W \epsilon_t. \tag{5}$$

Because the formula is already linear, there is no need to approximate it further.

3   Adaptive learning with the linear stock price formula

Equation (4) describes the relationship between the actual law of motion (ALM) of the current period stock price (l.h.s.) and its perceived law of motion (PLM) represented by the expectations on the next period stock price on the r.h.s. To be more concrete, substituting out m_{t+1} and d_{t+1} with their linear law of motions (3) and p_{t+1} with the perceived law of motion of the form:

$$p_{t+1} = \bar{p}^* + M_p^* \hat{s}_t + W_p^* \epsilon_{t+1}$$

we get the following mapping from the PLM to the ALM coefficients:

$$T \begin{pmatrix} \bar{p} \\ M_p \\ W_p \end{pmatrix} = \begin{pmatrix} \ln(A+B) \\ \left(M_m + \frac{A}{A+B} M_p^* + \frac{B}{A+B} M_d \right) M \\ \left(M_m + \frac{A}{A+B} M_p^* + \frac{B}{A+B} M_d \right) W \end{pmatrix} \tag{6}$$

$$A = \exp\left[\bar{m} + \bar{p}^* + 0.5(W_m + W_p^*)\Sigma(W_m + W_p^*)' \right],$$

$$B = \exp\left[\bar{m} + \bar{d} + 0.5(W_m + W_d)\Sigma(W_m + W_d)' \right].$$

Under rational expectations, the coefficients of the PLM coincide exactly with the ALM ones: $\bar{p}^{RE} = \bar{p}^*$, $M_p^{RE} = M_p^*$, $W_p^{RE} = W_p^*$. Therefore, they can be defined as a fixed point of the mapping (6).

In the adaptive learning case, agents do not know the values of $\bar{p}^*$, M_p^* and W_p^* and try to estimate them using observed values of p_t and $\hat{s}_t$. We assume that they use correctly specified model for the PLM:

$$p_t = \bar{p}^{AL} + M_p^{AL} \hat{s}_{t-1} + W_p^{AL} \epsilon_t. \tag{7}$$

In the baseline version of the study, agents learn only on $\bar{p}^{AL}$ and M_p^{AL}. We treat the standard deviation of the stochastic shock as known and set $W_p^{AL} = W_p^{RE}$. Later, we release this assumption and allow for learning on W_p^{AL} but it turns out that it has a negligible impact on stock prices.

As far as learning algorithms are concerned, we use the standard recursive least squares constant gain algorithm. It is a weighted version of the least squares procedure with exponentially declining weights for older observations. If $\phi_t = \begin{bmatrix} \bar{p}^{AL} & M_p^{AL} \end{bmatrix}'$ denotes vector of the estimated parameters in period t and R_t is covariance matrix of the estimates, then the algorithm can be described by the following recursion:

$$R_t = R_{t-1} + g\left(x_{t-1} x_{t-1}' - R_{t-1} \right), \tag{8}$$

$$\phi_t = \phi_{t-1} + g R_t^{-1} x_{t-1} \left(p_t - x_{t-1}' \phi_{t-1} \right), \tag{9}$$

where $x_t = \begin{bmatrix} 1 & \hat{s}_t \end{bmatrix}'$ represents vector of the observed explanatory variables in the regression (7) whereas the gain parameter g determines the decay rate of the weights for older

observations. Higher g puts more weight on the most recent observations in the learning process. As a consequence, the estimates ϕ_t fluctuate permanently around some mean levels instead of converging pointwise as in the decreasing gain schemes. We believe that this captures the permanent nature of learning of rapidly changing economic processes. To initialize the procedure, we first simulate the rational expectations version of the model for 200 periods and use the simulated data to calculate ϕ_0 and R_0 using ordinary least squares formulas.

If we assume that agents learn also on the standard deviation of the stochastic shock, we apply exactly the same procedure for estimating $\bar{p}$ and M_p. Then, we calculate W_p as square root of weighted average of the squared residuals from the regression. In other words, we treat ϵ_t like an error term in the standard linear regression. Obviously, such approach can be applied only to a model with one stochastic shock. With multiple shocks, the problem is more complicated due to the identification issue.

Under constant gain learning, it is possible that the adaptive learning coefficients deviate very far from their rational expectations counterparts. To exclude such extreme situations, a projection facility is employed. Whenever the difference in stock prices under adaptive learning and rational expectations exceeds some prespecified level, we proportionally change all the adaptive learning coefficients towards the rational expectations values by a fraction of 0.5. To be more precise, if:

$$\frac{p_t^{AL} - p_t^{RE}}{p_t^{RE}} > 1 \quad \text{or} \quad \frac{p_t^{AL} - p_t^{RE}}{p_t^{RE}} < -0.5,$$

then:

$$\phi_t = \phi_{t-1} - 0.5 \left(\phi_{t-1} - \phi^{RE} \right),$$

where ϕ_{RE} denotes the rational expectations values of ϕ.

4   Models for state variables dynamics

In the baseline version of the study, we use the model proposed by Jermann (1998) to describe dynamics of the state variables. This is a popular extension of the standard stochastic growth model with habits in the utility function and convex investment costs.

These two features are sufficient to generate reasonable equity premium. Below, we describe the basic version of the model.

4.1 Baseline model

The representative household is characterized by the constant relative risk aversion utility function with internal habits:

$$u(C_t, C_{t-1}) = \begin{cases} \dfrac{(C_t - \chi_c C_{t-1})^{1-\nu} - 1}{1 - \nu} & \text{for } \nu \neq 1, \\[2mm] \ln(C_t - \chi_c C_{t-1}) & \text{for } \nu = 1, \end{cases} \tag{10}$$

where C_t denotes consumption, ν is a curvature parameter and χ_c determines habit strength. Every period, the household maximizes its lifetime utility subject to the series of budget constrains of the form:

$$C_t = D_t + W_t, \tag{11}$$

where D_t denotes the dividends paid by the representative firm to the household and W_t represents the wage.

The firm combines capital C_t with labour L_t to produce output Y_t according to the constant return-to-scale technology:

$$Y_t = Z_t K_t^{\alpha} (A_t L_t)^{1-\alpha}, \tag{12}$$

where Z_t is the stochastic productivity shock, $A_t = \gamma A_{t-1}$ represents the labor-augmenting technical progress with the growth rate γ. Dynamics of the shock is described by the standard autoregression model:

$$\log Z_t = \rho_z \log Z_{t-1} + \sigma_z \epsilon_t, \quad \epsilon \sim N(0,1). \tag{13}$$

Capital depreciates at a constant rate δ and is increased by investment I_t. As a result, the law of motion for capital is given by:

$$K_t = \left[1 - \delta + \Phi\left(\frac{I_t}{K_t}\right) \right] K_{t-1}, \tag{14}$$

where function $\Phi(\cdot)$ represents the convex investment costs:

$$\Phi\left(\frac{I_t}{K_t}\right) = \frac{(\gamma - 1 + \delta)^{1/\xi}}{1 - 1/\xi}\left(\frac{I_t}{K_t}\right)^{1-1/\xi} + \frac{\gamma - 1 + \delta}{1 - \xi}. \tag{15}$$

The parameter ξ governs the function's convexity. Higher values of ξ mean lower investment costs. In the limit, we have: $\Phi\left(\frac{I_t}{K_t}\right) = \frac{I_t}{K_t}$ as $\xi \to \infty$.

Because of the technical progress, the variables need to be stationarized by A_t.[3] The stationarized variables are denoted with tildes. Summing up, the model consists of the following equations:

$$Q_t = \mathbb{E}_t\left[M_{t+1}\left(\alpha Z_{t+1}\left(\frac{\widetilde{K}_{t+1}}{\gamma}\right)^{\alpha - 1} - \frac{\gamma \widetilde{I}_{t+1}}{\widetilde{K}_{t+1}} + Q_{t+1}\left[1 - \delta + \Phi\left(\frac{\gamma \widetilde{I}_{t+1}}{\widetilde{K}_{t+1}}\right)\right]\right)\right], \tag{16}$$

$$\gamma K_t = \left[1 - \delta + \Phi\left(\frac{\gamma I_t}{K_t}\right)\right]K_{t-1}, \tag{17}$$

$$M_t = \frac{\widetilde{\beta}}{\gamma}\frac{MU_t}{MU_{t-1}} \tag{18}$$

$$MU_t = \left(\widetilde{C}_t - \frac{\chi_c}{\gamma}\widetilde{C}_{t-1}\right)^{-\nu} - \chi_c\frac{\widetilde{\beta}}{\gamma}\mathbb{E}_t\left[\left(\widetilde{C}_{t+1} - \frac{\chi_c}{\gamma}\widetilde{C}_t\right)^{-\nu}\right], \tag{19}$$

$$Q_t = (\gamma - 1 + \delta)^{-1/\xi}\left(\frac{\gamma \widetilde{I}_t}{\widetilde{K}_t}\right)^{1/\xi}, \tag{20}$$

$$\widetilde{Y}_t = Z_t\left(\frac{\widetilde{K}_t}{\gamma}\right)^{\alpha}, \tag{21}$$

$$\widetilde{Y}_t = \widetilde{C}_t + \widetilde{I}_t, \tag{22}$$

$$\widetilde{D}_t = \alpha\widetilde{Y}_t - \widetilde{I}_t, \tag{23}$$

$$Z_t = \rho_z Z_{t-1} + \sigma_z \epsilon_t. \tag{24}$$

Q_t is the Lagrange multiplier for the firm's optimization problem. It can be interpreted as Tobin's marginal q, MU_t denotes the household's marginal utility from consumption whereas $\widetilde{\beta}$ is the discount coefficient in the stationarized version of the model. Because labor does not enter the utility function it is normalized to 1.

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[3]Capital is the exception here because in period t it is divided by A_{t-1}.

Table 1: Parametrization of the models

α	$\tilde{\beta}$	δ	γ	σ	ρ	ν	ξ	χ_c	χ_h
				Baseline model					
0.36	0.99	0.025	1.005	0.01	0.99	5	0.23	0.82	—
			No habits, no investment costs						
0.36	0.99	0.025	1.005	0.01	0.99	5	$+\infty$	0	—
				Slow-moving habits					
0.36	0.991	0.025	1.007	0.01	0.99	1	0.24	—	0.85

4.2 Other models

For the robustness check, we also consider two modifications of the previously described model: the standard stochastic growth model without habits ($\chi_c = 0$) and investment costs ($\xi = +\infty$) as well as the slow-moving habits model proposed by Jaccard (2014). In the latter model, the household values consumption relative to a habit level H_t, so the utility function is given by:

$$u(C_t, H_t) = \begin{cases} \dfrac{(C_t - H_t)^{1-\nu} - 1}{1 - \nu} & \text{for } \nu \neq 1, \\ \ln(C_t - H_t) & \text{for } \nu = 1, \end{cases} \tag{25}$$

where stationarized habit stock dynamics follows the moving average representation:

$$\gamma \widetilde{H}_t = \chi_h \widetilde{H}_{t-1} + (1 - \chi_h)\widetilde{C}_{t-1}. \tag{26}$$

4.3 Parametrizations

For the parametrization of the baseline model, we use the values taken from Jermann (1998) where a period in the model corresponds to a quarter. We also utilize the same set of parameters for the standard stochastic growth model without habits and investment costs. As far as the slow-moving habit model is concerned, we follow Jaccard (2014). The calibrations are summarized in table 1.

4.4 Additional remarks

For every model, we find the loglinear solution of the form (3) and calculate the risk-free rate using formula (5). We also calculate the stock price coefficients under rational expectations as the fixed point of the mapping (6) as well as for the adaptive learning scheme using the constant gain algorithm (8). For the accuracy check purposes, we also solve the models with the third order perturbation method with variables in logs and the standard asset pricing formulas.[4] We use Dynare to find the solutions of the models.

5 Results

5.1 Results for the baseline model

The simulated moments for the baseline model under the different expectation schemes are reported in table 2. In panel A, we have the moments of the financial variables. The adaptive learning mechanism significantly increases the mean stock return which rises from 2.28% per quarter to 2.66%. As a result, the equity premium also increases, from 1.83% to 2.26%. At the same time, the stock price volatility grows only by 0.4 p.p. which is not much in relative terms. However, the volatility considerably exceeds the level observed in the data. The magnitude of these effects strongly depends on the learning coefficient g. For example, if $g = 0.05$, then the equity premium jumps to 3.17%. Adaptive learning also increases autocorrelation of the dividend-price ratio. In the rational expectations version of the model, the autocorrelation coefficient equals 0.8, whereas in the data it exceeds 0.98. Under the adaptive learning scheme, the coefficient is about 0.93.

Panel B shows that the adaptive learning mechanism is also able to generate significant amount of the equity premium predictability. Under rational expectations, median R_4^2 from the one-year-ahead predictive regression is about 0.008, whereas in the data it is about 0.1. Moreover, coefficient $\hat{b}_4$ is statistically significant in less than 50% of simulations. On the other hand, under the adaptive learning scheme, R_4^2 rises to 0.03 and the fraction of the statistically significant predictive coefficients is close to 70%. Because of the higher autocorrelation of the dividend-price series, the effect becomes

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[4]The second-order solution is calculated for the slow-moving habits model because the third order approximation is numerically unstable during simulations.

Table 2: Results for the baseline model

Moment	Data	RE	RE 3rd ord.	AL $g = 0.03$	AL $g = 0.05$	AL W_p learn	AL no $\bar{p}$ learn
A. Moments of the financial variables							
$\mathbb{E}(R)$	1.58	2.28	2.23	2.66	3.56	2.59	2.32
$\mathbb{D}(R)$	6.30	12.89	12.63	13.19	16.20	13.05	12.62
corr(R)	0.38	-0.14	-0.14	-0.06	-0.02	-0.06	-0.09
$\mathbb{E}(R - R_f)$	1.39	1.83	1.77	2.26	3.17	2.22	1.93
$\mathbb{D}(R - R_f)$	6.29	12.17	11.92	12.73	16.06	12.63	12.07
corr$(R - R_f)$	0.37	0.00	-0.01	0.08	0.10	0.09	0.07
$\mathbb{E}(DP)$	3.46	3.81	3.73	4.39	6.07	4.24	3.87
$\mathbb{D}(DP)$	1.43	0.84	0.82	2.00	3.91	1.92	1.12
corr(DP)	0.986	0.80	0.80	0.93	0.93	0.93	0.87
B. Predictability of the equity premium							
$\hat{b}_4$	0.051	0.020	0.020	0.045	0.082	0.043	0.030
sig $\hat{b}_4$	—	44.7	45.9	69.2	77.9	68.3	61.1
R_1^2	0.041	0.002	0.002	0.006	0.016	0.006	0.003
R_4^2	0.102	0.009	0.009	0.030	0.059	0.026	0.017
R_{16}^2	0.268	0.023	0.022	0.095	0.158	0.091	0.057
$\mathbb{D}(EEP)$	—	0.08	0.12	3.55	5.18	2.13	3.10
C. Volatility of the macro variables							
$\mathbb{D}(\Delta Y)$	0.01	0.01	0.01	0.01			
$\mathbb{D}(\Delta C)/\mathbb{D}(\Delta Y)$	0.51	0.49	0.49	0.49			
$\mathbb{D}(\Delta I)/\mathbb{D}(\Delta Y)$	5.19	2.85	2.81	2.85			
D. Projection facility							
frac	—	—	—	27.1	69.5	25.5	2.3
av. num.	—	—	—	1.8	3.6	1.8	1.0

The table shows medians for 1000 simulation with 260 quarterly observations each. All results are for quarterly data. In the second column, the empirical characteristics are reported based on the US data from Robert Shiller's website (stock returns, risk-free rate and dividends 1Q1948–4Q2012) and from the textbook of DeJong and Dave (2011) (output, consumption and investment 1Q1948–1Q2010).

Panel B: Predictability of the equity premium examined by running regressions of the h-period-ahead excess returns on the log dividend-price ratio: $\prod_{i=1}^{h} EP_{t+i} = b_0 + b_h \log DP_t$, where $EP_t = R_t - R_{f,t-1}$, R — stock return, R_f — risk-free rate, $DP_t = (D_{t-3} + D_{t-2} + D_{t-1} + D_t)/P_t$. The estimates of the regression coefficient b_4 for the one-year-ahead excess returns and the determination coefficients R^2 for $h = 1, 4, 16$ quarters are reported together with the fraction of simulations with the statistically significant estimates of b_4 at the 5% significance level (sig $\hat{b}_4$); $EEP_t = \mathbb{E}_t R_{t+1} - R_{f,t}$ — expected equity premium.

Panel C: Volatility of the macro variables is calculated for their quarterly growth rates denoted by Δ.

Panel D: frac – fraction of simulations where the projection facility was used; av. num. – average number of periods in a simulation when the projection facility was used.

even stronger as the regression horizon h increases. Furthermore, predictability rises when the learning coefficient g grows. For example, R_4^2 for the one-year-ahead excess returns is now 0.059 and the fraction of the significant predictive coefficients reaches almost 78%.

The last row of the panel B documents that the increase in the equity premium predictability under adaptive learning is associated with the substantial rise in the volatility of the expected equity premium, which is virtually constant under rational expectations. The variation of the expected excess returns is well in line with the seminal finding of Fama and French (1989).

Finally, panel D documents that the projection facility is used in about 25% of simulations, less than two times per simulation on average, for the adaptive learning model with $g = 0.03$. Therefore, this mechanism could not seriously influence the presented results, especially as medians are reported. Obviously, for the higher value of g, the projection facility is called more often.

5.2   Predictability of the equity premium with longer time-series

The rise of the equity premium predictability under adaptive learning becomes even more striking when longer time series are concerned. Table 3 contains the predictability statistics for series with 1000 observations. Under rational expectations, R_4^2 for the predictive regression with the one-year-ahead excess returns equals only 0.002 and the fraction of the statistically significant predictive coefficients is about 40%. Under adaptive learning, R_4^2 is about one order of magnitude higher and the fraction of the significant coefficients exceeds 90%. The differences are even greater for the four-year-ahead excess returns. For example, R_{16}^2 rises from 0.5% to 8.4%. However, these results partially comes from the rise in autocorrelation of the dividend-price ratio which exceeds 0.96 under the adaptive learning scheme and is considerably larger than for the series with 260 observations.

5.3   Other specifications of the perceived law of motion

As a second exercise, we compare the results for the adaptive learning models with different information sets. In the baseline scheme, we assume that the agents know the

Table 3: Results for the baseline model with $n = 1000$ observations

Moment	Data	RE	AL $g = 0.03$
A. Predictability of the equity premium			
$\hat{b}_4$	0.051	0.009	0.050
sig $\hat{b}_4$	—	40.8	90.5
R^2_1	0.041	0.001	0.006
R^2_4	0.102	0.002	0.026
R^2_{16}	0.268	0.005	0.084
$\mathbb{D}(EEP)$	—	0.08	4.16
B. Projection facility			
frac	—	—	70.7
av. num.	—	—	2.7

The table shows median results for 1000 simulation with 1000 quarterly observations each. All results are for quarterly data. In the second column, the empirical characteristics are reported based on the US data from Robert Shiller's website (stock returns, risk-free rate and dividends 1Q1948–4Q2012) and from the textbook of DeJong and Dave (2011) (output, consumption and investment 1Q1948–1Q2010). Panel A: Predictability of the equity premium examined by running regressions of the h-period-ahead excess returns on the log dividend-price ratio: $\prod_{i=1}^{h} EP_{t+i} = b_0 + b_h \log DP_t$, where $EP_t = R_t - R_{f,t-1}$, R — stock return, R_f — risk-free rate, $DP_t = (D_{t-3} + D_{t-2} + D_{t-1} + D_t)/P_t$. The estimates of the regression coefficient b_4 for the one-year-ahead excess returns and the determination coefficients R^2 for $h = 1, 4, 16$ quarters are reported together with the fraction of simulations with the statistically significant estimates of b_4 at the 5% significance level (sig $\hat{b}_4$); $EEP_t = \mathbb{E}_t R_{t+1} - R_{f,t}$ — expected equity premium.

Panel B: frac – fraction of simulations where the projection facility was used; av. num. – average number of periods in a simulation when the projection facility was used.

conditional standard deviation W_p of stock prices and estimate $\bar{p}$ and M_p. Here, we consider two other assumptions: first, that agents learn also on W_p, and second, that they learn only on M_p (no $\bar{p}$ learn). As far as the financial moments are concerned, learning on W_p does not affect the results significantly. In fact, they are even slightly closer to the rational expectations version compared to the baseline learning scheme. If we turn to learning on M_p only, the results are generally close to the rational expectations ones. Therefore, the observed effects of adaptive learning can be attributed mainly to learning on the mean stock price $\bar{p}$.

However, when the equity premium predictability is concerned, learning on M_p plays also a role. Compared with the rational expectations case, R_4^2 rises to 0.017 and the percentage of the significant predictive coefficients increases by 15 p.p.

5.4 Accuracy check

In this subsection, we examine accuracy of our approximate stock price formula by comparing the simulated characteristics of asset prices with their counterparts calculated for the model approximated with the third order perturbation method. Generally, the differences between the two approaches are tiny. For our solution method, the mean stock return is only 0.05 p.p. higher than for the benchmark. The difference in volatility of the stock returns is 0.26 p.p. Similar discrepancies are observed for the equity premium. As far as predictability of the equity premium is concerned, the results for both method are virtually the same. These results clearly show that our stock price approximation method provides sufficient accuracy, at least for the analyzed model.

6 Robustness checks

6.1 Standard stochastic growth model

Table 4 contains the results for the standard stochastic growth model. If we look at the baseline learning scheme, we can see that the adaptive learning mechanism also increases the stock returns and the equity premium, but the effects in absolute terms are rather tiny. For example, the equity premium rises from 0.02% to 0.07%. However, this is still a considerable rise when relative changes are concerned. Similarly to the baseline model,

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the adaptive learning mechanism also increases the equity premium predictability. For the one-year-ahead excess returns, the fraction of the statistically significant coefficients grows from 78% to 86.3% and R_4^2 rises from 0.064 to 0.101.

As far as the other information sets are concerned, we can notice that learning on the conditional standard deviation W_p also does not change the results much. However, if agents learn on M_p only then the results on the asset returns are similar to the rational expectations case, but the characteristics describing the equity premium predictability lay midway between the results for the rational expectations version and the baseline adaptive learning case.

To summarize, the results confirm the findings for the Jermann model, that the decisive impact on stock return moments has learning on the mean stock price $\bar{p}$, but contribution of unknown M_p is also important for the equity premium predictability.

However, one should be careful about magnitude of the discussed effects because they may be in some part a byproduct of our stock price approximation. Unfortunately, we observe some non-negligible differences between our solution method and the benchmark one. Under the third order perturbation, the equity premium is virtually 0, whereas it is 0.02% for our method. And similarly, R_4^2 from the one-year ahead predictive regression is 0.039 under the benchmark and 0.064 for our approach.

6.2 Slow-moving habits model

The simulated moments of the slow-moving habits model are shown in table 5. The general conclusions are similar to those for the both previously discussed models. The adaptive learning mechanism increases the equity premium (from 1.1% to 1.4%) and its volatility (from 9.95% to 10.85%). It also rises the equity premium predictability. For the one-year-ahead excess returns, the fraction of the significant predictive coefficients grows from 45.7% to 71.4% and R_4^2 soars from 0.008 to 0.037. And again, learning on the conditional standard deviation W_p does not change the results at all. Instead, learning on the mean stock price $\bar{p}$ explains the majority of the observed learning effects.

Table 4: Results for the no habits, no investment costs model

Moment	Data	RE	RE 3rd ord.	AL $g = 0.06$	AL W_p learn	AL no $\bar{p}$ learn
A. Moments of the financial variables						
$\mathbb{E}(R)$	1.58	1.46	1.44	1.52	1.53	1.46
$\mathbb{D}(R)$	6.30	0.23	0.18	1.09	1.07	0.24
$\text{corr}(R)$	0.38	0.72	0.98	0.33	0.33	0.78
$\mathbb{E}(R - R_f)$	1.39	0.02	0.00	0.07	0.08	0.02
$\mathbb{D}(R - R_f)$	6.29	0.14	0.04	1.05	1.06	0.13
$\text{corr}(R - R_f)$	0.37	0.02	0.01	0.31	0.31	0.15
$\mathbb{E}(DP)$	3.46	3.76	3.71	3.94	3.97	3.75
$\mathbb{D}(DP)$	1.43	0.39	0.39	0.69	0.69	0.38
$\text{corr}(DP)$	0.986	0.998	0.998	0.997	0.997	0.998
B. Predictability of the equity premium						
$\hat{b}_4$	0.051	0.001	0.000	0.007	0.007	0.000
sig $\hat{b}_4$	—	78.0	75.7	86.3	86.3	83.5
R_1^2	0.041	0.017	0.010	0.039	0.038	0.026
R_4^2	0.102	0.064	0.039	0.101	0.102	0.082
R_{16}^2	0.268	0.222	0.131	0.261	0.276	0.254
$\mathbb{D}(EEP)$	—	0.01	0.00	1.01	0.82	0.07
C. Volatility of the macro variables						
$\mathbb{D}(\Delta Y)$	0.01	0.01	0.01	0.01		
$\mathbb{D}(\Delta C)/\mathbb{D}(\Delta Y)$	0.51	0.77	0.77	0.77		
$\mathbb{D}(\Delta I)/\mathbb{D}(\Delta Y)$	5.19	1.62	1.61	1.62		
D. Projection facility						
frac	—	—	—	11.2	10.9	0.0
av. num.	—	—	—	2.7	2.8	—

The table shows medians for 1000 simulation with 260 quarterly observations each. All results are for quarterly data. In the second column, the empirical characteristics are reported based on the US data from Robert Shiller's website (stock returns, risk-free rate and dividends 1Q1948–4Q2012) and from the textbook of DeJong and Dave (2011) (output, consumption and investment 1Q1948–1Q2010).

Panel B: Predictability of the equity premium examined by running regressions of the h-period-ahead excess returns on the log dividend-price ratio: $\prod_{i=1}^{h} EP_{t+i} = b_0 + b_h \log DP_t$, where $EP_t = R_t - R_{f,t-1}$, R — stock return, R_f — risk-free rate, $DP_t = (D_{t-3} + D_{t-2} + D_{t-1} + D_t)/P_t$. The estimates of the regression coefficient b_4 for the one-year-ahead excess returns and the determination coefficients R^2 for $h = 1, 4, 16$ quarters are reported together with the fraction of simulations with the statistically significant estimates of b_4 at the 5% significance level (sig $\hat{b}_4$); $EEP_t = \mathbb{E}_t R_{t+1} - R_{f,t}$ — expected equity premium.

Panel C: Volatility of the macro variables is calculated for their quarterly growth rates denoted by Δ.

Panel D: frac – fraction of simulations where the projection facility was used; av. num. – average number of periods in a simulation when the projection facility was used.

Table 5: Results for the slow-moving habits model

Moment	Data	RE	RE 2nd ord.	AL $g = 0.03$	AL W_p learn	AL no $\bar{p}$ learn
A. Moments of the financial variables						
$\mathbb{E}(R)$	1.58	2.17	2.11	2.40	2.36	2.18
$\mathbb{D}(R)$	6.30	10.19	10.09	10.99	10.92	9.95
corr(R)	0.38	-0.02	-0.02	0.09	0.09	0.06
$\mathbb{E}(R - R_f)$	1.39	1.10	1.03	1.40	1.38	1.19
$\mathbb{D}(R - R_f)$	6.29	9.95	9.84	10.85	10.81	9.79
corr$(R - R_f)$	0.37	-0.00	-0.00	0.11	0.12	0.09
$\mathbb{E}(DP)$	3.46	3.79	3.58	4.25	4.16	3.90
$\mathbb{D}(DP)$	1.43	1.34	1.31	2.47	2.39	1.60
corr(DP)	0.986	0.945	0.944	0.971	0.971	0.962
B. Predictability of the equity premium						
$\hat{b}_4$	0.051	0.011	0.013	0.043	0.043	0.024
sig $\hat{b}_4$	—	45.7	47.2	71.4	71.5	61.2
R_1^2	0.041	0.002	0.003	0.010	0.009	0.004
R_4^2	0.102	0.008	0.010	0.037	0.038	0.019
R_{16}^2	0.268	0.026	0.030	0.139	0.139	0.064
$\mathbb{D}(EEP)$	—	0.08	0.02	3.58	2.02	2.80
C. Volatility of the macro variables						
$\mathbb{D}(\Delta Y)$	0.01	0.01	0.01			
$\mathbb{D}(\Delta C)/\mathbb{D}(\Delta Y)$	0.51	0.48	0.50			
$\mathbb{D}(\Delta I)/\mathbb{D}(\Delta Y)$	5.19	2.36	2.34			
D. Projection facility						
frac	—	—	—	27.1	27.3	3.6
av. num.	—	—	—	1.3	1.3	1.1

The table shows medians for 1000 simulation with 260 quarterly observations each. All results are for quarterly data. In the second column, the empirical characteristics are reported based on the US data from Robert Shiller's website (stock returns, risk-free rate and dividends 1Q1948–4Q2012) and from the textbook of DeJong and Dave (2011) (output, consumption and investment 1Q1948–1Q2010).

Panel B: Predictability of the equity premium examined by running regressions of the h-period-ahead excess returns on the log dividend-price ratio: $\prod_{i=1}^h EP_{t+i} = b_0 + b_h \log DP_t$, where $EP_t = R_t - R_{f,t-1}$, R — stock return, R_f — risk-free rate, $DP_t = (D_{t-3} + D_{t-2} + D_{t-1} + D_t)/P_t$. The estimates of the regression coefficient b_4 for the one-year-ahead excess returns and the determination coefficients R^2 for $h = 1, 4, 16$ quarters are reported together with the fraction of simulations with the statistically significant estimates of b_4 at the 5% significance level (sig $\hat{b}_4$); $EEP_t = \mathbb{E}_t R_{t+1} - R_{f,t}$ — expected equity premium.

Panel C: Volatility of the macro variables is calculated for their quarterly growth rates denoted by Δ.

Panel D: frac – fraction of simulations where the projection facility was used; av. num. – average number of periods in a simulation when the projection facility was used.

6.3 State variables learning in the baseline model

In our previous considerations, we always assume that the agents hold rational expectations for all variables but the stock price. Here, we reverse this assumption by letting the agents learn on the macroeconomic state variables instead of stock prices. We carry out the analysis using the setting proposed by Jermann (1998), for which the state variables are capital, the technological shock and the previous-period consumption. Obviously, the state variables learning mechanism affects dynamics of all other variables in the model as well. We employ the same constant gain recursive least squares scheme, although we set different values of the learning coefficient g. The details of the state variables learning version of the model are given in the technical appendix.

Table 6 summarizes the results for two values of the learning coefficient: $g = 0.015$ and $g = 0.025$. As far as the returns are concerned, the mean stock return increases from 2.28% to 2.53% for the lower value of g and to 3.13% for the higher one. However, the risk-free rate rises too, from 0.44% to 1.01% and 0.96%, respectively. As a result, the effect for the equity premium is ambiguous. For $g = 0.015$, it is lower than in the rational expectations case, whereas for $g = 0.025$, it is slightly higher. The adaptive learning assumption also leads to the increase in volatility of the variables, especially the risk-free rate. It jumps from 4.11% to 8.41% and 11.03% respectively, which is far from the stylized facts. This is also the case for the strongly negative autocorrelation of the risk-free rate.

The results from panel B clearly show that replacing the rational expectations hypothesis with the adaptive learning assumption has a negligible impact on the equity premium predictability. In fact, under adaptive learning it is slightly smaller than under rational expectations. The fraction of the statistically significant predictive coefficients for the one-year-ahead excess returns drops from 45% to less than 35%. Similarly, R_4^2 becomes slightly lower, which mainly results from the drop in the dividend-price auto-correlation coefficient.

Finally, it should also be pointed out that the adaptive learning mechanism almost does not influence the macroeconomic variables dynamics. The relative volatility of consumption remains unchanged, whereas for investments it increases only by a small margin, from 2.82 to 2.92 and 2.95, respectively.

Table 6: Results for the baseline model with state variables learning

Moment	Data	RE	AL $g = 0.015$	AL $g = 0.025$
A. Moments of the financial variables				
$\mathbb{E}(R)$	1.58	2.28	2.53	3.13
$\mathbb{D}(R)$	6.30	12.89	14.72	18.51
$\mathrm{corr}(R)$	0.38	-0.14	-0.23	-0.27
$\mathbb{E}(R_f)$	0.20	0.44	1.01	0.96
$\mathbb{D}(R_f)$	0.85	4.11	8.41	11.03
$\mathrm{corr}(R_f)$	0.33	0.68	0.22	0.14
$\mathbb{E}(R - R_f)$	1.39	1.83	1.41	2.02
$\mathbb{D}(R - R_f)$	6.29	12.17	11.60	14.37
$\mathrm{corr}(R - R_f)$	0.37	0.00	-0.00	-0.01
$\mathbb{E}(DP)$	3.46	3.81	3.81	3.90
$\mathbb{D}(DP)$	1.43	0.84	0.89	1.02
$\mathrm{corr}(DP)$	0.986	0.80	0.76	0.70
B. Predictability of the equity premium				
$\hat{b}_4$	0.051	0.020	0.006	0.007
sig $\hat{b}_4$	—	44.7	38.0	34.8
R_1^2	0.041	0.002	0.002	0.002
R_4^2	0.102	0.009	0.007	0.006
R_{16}^2	0.268	0.023	0.022	0.017
$\mathbb{D}(EEP)$	—	0.08	10.54	14.51
C. Volatility of the macro variables				
$\mathbb{D}(\Delta Y)$	0.01	0.01	0.01	0.01
$\mathbb{D}(\Delta C)/\mathbb{D}(\Delta Y)$	0.51	0.49	0.49	0.49
$\mathbb{D}(\Delta I)/\mathbb{D}(\Delta Y)$	5.19	2.85	2.92	2.97
D. Projection facility				
frac	—	—	0.0	0.0
av. num.	—	—	—	—

The table shows medians for 1000 simulation with 260 quarterly observations each. All results are for quarterly data. In the second column, the empirical characteristics are reported based on the US data from Robert Shiller's website (stock returns, risk-free rate and dividends 1Q1948–4Q2012) and from the textbook of DeJong and Dave (2011) (output, consumption and investment 1Q1948–1Q2010).

Panel B: Predictability of the equity premium examined by running regressions of the h-period-ahead excess returns on the log dividend-price ratio: $\prod_{i=1}^{h} EP_{t+i} = b_0 + b_h \log DP_t$, where $EP_t = R_t - R_{f,t-1}$, R — stock return, R_f — risk-free rate, $DP_t = (D_{t-3} + D_{t-2} + D_{t-1} + D_t)/P_t$. The estimates of the regression coefficient b_4 for the one-year-ahead excess returns and the determination coefficients R^2 for $h = 1, 4, 16$ quarters are reported together with the fraction of simulations with the statistically significant estimates of b_4 at the 5% significance level (sig $\hat{b}_4$); $EEP_t = \mathbb{E}_t R_{t+1} - R_{f,t}$ — expected equity premium.

Panel C: Volatility of the macro variables is calculated for their quarterly growth rates denoted by Δ.

Panel D: frac – fraction of simulations where the projection facility was used; av. num. – average number of periods in a simulation when the projection facility was used.

Summing up, the presented results prove that introducing the adaptive learning scheme for the macroeconomic state variables results in the significant changes in the asset price characteristics, especially for the risk-free rate. However, in most cases, it worsens the model's ability to match the basic stylized facts for asset prices partially offsetting the impact of stock price learning.

7 Conclusion

We study the quantitative consequences of replacing the rational expectations assumption with the self-referential adaptive learning scheme in a few general equilibrium models with the production sector focusing on the first and second moments of the stock returns, the equity premium and its predictability. We introduce a novel, linearized version of the loglinear-lognormal stock price formula which allows us to employ the standard linear learning schemes.

The results basically confirm the findings of Carceles-Poveda and Giannitsarou (2008), that in the standard stochastic growth model the adaptive learning can neither generate the reasonable equity premium level nor explain the excess return predictability. However, in more comprehensive models, it can successfully amplify effects generated by other mechanisms. In particular, in our baseline model proposed by Jermann (1998), it significantly increases the stock returns, the equity premium as well as its predictability. It also rises the volatility of the stock returns, although the effect is considerably smaller.

Because the adaptive learning mechanism can improve asset price implications of a model, it allows a researcher to employ a less extreme parametrization to match the asset price characteristics. This usually improves a model's fit in other important areas. For example, in the analysed Jermann model introducing adaptive learning allows to set lower values of the habit strength or the investment cost parameters to match the equity premium. As a result, volatility of the risk-free rate is also lower and closer to the data. However, this might not be the case if one extends the adaptive learning assumption to the macroeconomic state variables, which can seriously increase fluctuations of the risk-free rate.

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